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SENSITIVITY ANALYSIS RESULTS ON THE SEPARATION PROBLEM OF BOLTED STEEL COLUMN-TO-COLUMN CONNECTIONS†

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Abstract—The present paper deals with the sensitivity analysis of bolted steel column-to-column connections, by taking into account the development of separation phenomena on the joint endplates. A variational inequality and quadratic programming approach is first proposed to the investigation of the separation problem on such bolted steel connections. Applying the classical unilateral contact law to describe in a quasi-static way the separation process along the surface between the splice plates, the continuous problem can be formulated either as a variational inequality or equivalently as a quadratic programming problem. Then, by means of an appropriate finite element discretization scheme, the discrete problem is formulated as a quadratic optimization problem with inequality constraints. In order to investigate the variation of the structural response of the connection under consideration due to the variation of critical design parameters, the sensitivity analysis problem is formulated; the latter is a quadratic programming problem where design parameters appear only in the quadratic term. This problem can be effectively treated numerically by means of an appropriate quadratic optimization algorithm. The applicability and the effectiveness of the method are illustrated by means of two numerical applications.

1. INTRODUCTION

Bolted plate connections are used in structural steelwork to transmit internal forces between adjacent structural elements. Such joints applied for instance as column-to-base, column-to-column and beam-to-column connections are nowadays extensively used in any possible combination in steel structures. It is therefore, obvious that any improvement on the analysis of such connections would ameliorate the existing design and safety criteria dictated by steel construction codes.

Classical methods for the analysis and design of bolted steel connections often disregard splice plates' deformability by assuming "complete contact" between the splice plates. The structural behaviour of steel connections is in this case investigated *a priori*, using the aforementioned hypothesis of complete contact between the bolted steel endplates. A direct result of the latter assumption is that compression forces are absorbed by the plates in contact, whereas possible tension forces are transmitted by the bolts. However, due to the contradition between the previously mentioned simplifying assumption of complete contact between the splice plates and the evidence gained by laboratory testing and structural steelwork practice, the problem under investigation attracted the interest of a plethora of researchers working on this subject, either theoretically/numerically or experimentally (see e.g. Kato and McGuire, 1973; Paker and Morris, 1977; Chen and Patel, 1981; Raffa and Strona, 1984; Thomopoulos, 1985; Chen and Lui, 1986). In these studies, both the finite

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element and the boundary element methods among others have been applied to the simulation of the structural behaviour of steel column connections. A result of these studies was the ascertainment of the appearance of detachment (i.e. separation) phenomena on the adjacent fronts of the bolted endplates. Since the aforementioned phenomenon is of nonlinear nature and the contact or separation zones on the steel endplates are not *a priori* known, the application of the classical structural analysis methods to solve the problem at hand may lead to erroneous results and must therefore be used with care. Among the models recently proposed to simulate the response of steel connections, those combining the methods of nonsmooth mechanics (Panagiotopoulos, 1985; Moreau and Panagiotopoulos, 1988; and Moreau *et al.*, 1988) with finite element discretization schemes are worth mentioning here (cf. e.g. Abdalla, 1988; Abdalla and Stavroulakis, 1989; Baniotopoulos *et al.*, 1992; and Baniotopoulos and Abdalla, 1993). Applying such simulation models, the additional high nonlinearity produced by the development of the separation phenomenon can be taken into account as well. Within such a theoretical framework, the problem at hand can be studied by taking into account the possibility of development of local separation phenomena on the connection endplates. In particular, these separation phenomena can be completely described mathematically in both static and dynamic problems by the Signorini unilateral contact law, theoretically investigated by Fichera (1972); such an approach leads to a variational inequality or equivalently to a quadratic programming formulation of the problem. Indeed, variational inequalities take into account the exact nature of the unilateral contact nonlinearity. Such a treatment of the problem has, among others, both the advantages of exact determination of the active contact and separation zones between the contact fronts, and of exact evaluation of the loss of strength of the bolted steel plate connection due to the development of the separation phenomenon for a given loading (without any incremental procedure).

Concerning the theoretical treatment of the problem, the continuous problem being a typical boundary value problem (B.V.P.) is first formulated as a variational inequality problem with respect to displacements which express, from the standpoint of mechanics, the principle of virtual work in inequality form for the steel connection at the state of equilibrium. Such a formulation permits the derivation of the principle of minimum potential energy of the steel connection at the state of equilibrium in the form of a quadratic optimization problem, which involves a quadratic energy function coupled by inequality kinematic constraints (cf. e.g. Maier, 1968, 1973, and Panagiotopoulos, 1975, 1976). Applying an appropriately chosen finite element discretization scheme, the formulated problem can be equivalently put in the form of a discrete quadratic optimization problem. It is worth noting that the latter formulation seems very convenient because numerous quadratic optimization algorithms are nowadays available for the numerical treatment of the problem. It is also worth noting that a dual approach can also be employed to the mathematical formulation of the problem under consideration. In this case, the variational inequality problem with respect to stresses expresses, from the standpoint of mechanics, the principle of complementary virtual work in inequality form and the respective quadratic optimization problem, the principle of minimum complementary energy for the steel connection at the state of equilibrium.

In order to obtain sensitivity analysis results of the structural response of such steel connections, the unified approach for sensitivity analyses of unilateral problems for discrete and continuous elastic structures proposed by Bendsøe *et al.* (1985) is herein applied. Within this theoretical framework, directional derivatives with respect to variations of the variables appearing in the quadratic term of a similar quadratic optimization problem have to be defined. By means of such an analysis, the influence of small variations of the design parameters, as is for example the splice plate thickness, to the overall structural response of the column splices can be investigated. As shown by Bendsøe and Sokolowski (1988), this problem is similar to the sensitivity elastoplastic problem with the only difference that in the latter, design parameters appear in both the quadratic and the linear term.

The formulated quadratic optimization problems can be effectively treated numerically by employing a quadratic optimization method and, in particular, the Hildreth–d'Esopo algorithm which has been described in detail by Künzi and Krelle (1962). This solution

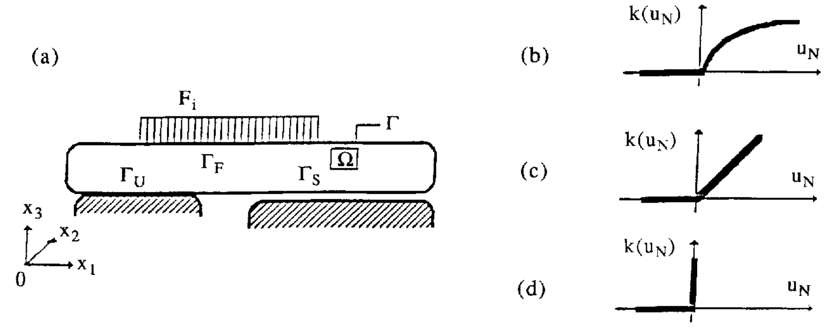

Fig. 1. On the continuous problem and the forms of $k(\cdot)$.

procedure is easily programmable and computationally efficient for the numerical treatment of the problem under consideration. The range of the applicability and the effectiveness of the proposed method are illustrated in the last part of the paper by means of two numerical applications.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

2.1. *The continuous problem*

An elastic body Ω with boundary Γ made up of three nonoverlapping parts Γ_U, Γ_F and Γ_S is considered in an orthogonal cartesian system $0x_1x_2x_3$. On Γ_U (respectively Γ_F) the displacements (respectively the surface forces) have given values U_i (respectively F_i), whereas on Γ_S, unilateral contact boundary conditions hold (Fig. 1a). On Γ_S, frictionless contact type is assumed and, in addition, as positive normal direction, the one directed outwards of the boundary is taken into consideration. The unilateral contact conditions with respect to an elastic support are expressed in the following form:

$$\text{if } u_N < 0 \quad \text{then} \quad S_N = 0 \tag{1}$$

$$\text{if } u_N \geqslant 0 \quad \text{then} \quad S_N - k(u_N) = 0, \tag{2}$$

where u_N (respectively S_N) denotes the normal (with respect to the boundary) displacements (respectively reaction forces) on Γ_S and $k(u_N)$ is a nondecreasing function. These relations are illustrated in Fig. 1b, whereas Fig. 1c and d correspond respectively to unilateral contact with a linearly elastic and with a rigid support.

Assuming that strains and displacements are small, the problem under consideration consists of the equation of equilibrium, the compatibility relations, the constitutive law relating stresses to strains and the boundary conditions holding on the boundary Γ. We define a field X^* of strains and displacements as being kinematically admissible if it satisfies the compatibility relations and the kinematical boundary conditions on Γ_U and on Γ_S. Volume forces are denoted by p_i, and actual strains and displacements at the position of equilibrium by ε_{ij} and u_i, respectively. The differences $(\varepsilon^*_{ij}-\varepsilon_{ij})$ and $(u^*_i-u_i)$ represent the kinematically admissible variations of the respective variables. The stress field obtained from ε^*_{ij} by means of the elasticity law is denoted by σ^*_{ij}. Splitting u^*_N into its positive and negative parts defined by the forms

$$u^*_{N+} = \frac{u^*_N + |u^*_N|}{2} \tag{3}$$

and

$$u^*_{N-} = \frac{-u^*_N + |u^*_N|}{2}, \tag{4}$$

which are non-negative quantities, the variational equality

$$\int_\Omega \sigma^*_{ij}(\varepsilon^*_{ij} - \varepsilon_{ij})\, d\Omega = \int_\Omega p_i(u^*_i - u_i)\, d\Omega + \int_{\Gamma_S} S_{Ni}(u^*_{Ni} - u_{Ni})\, d\Gamma + \int_{\Gamma_F} F_i(u^*_i - u_i)\, d\Gamma \quad \forall u^*_i \in X^*, \tag{5}$$

expressing that the virtual work of the internal forces is equal to the virtual work of the external forces for the body Ω, combined to the inequality

$$\int_{\Gamma_S} (S_{Ni}(u^*_{Ni} - u_{Ni}) + k(u_{N+})(u^*_{N+} - u_{N+}))\, d\Gamma \geqslant 0 \quad \forall u^*_{Ni} \in X^*, \tag{6}$$

holding on Γ_S, yields by means of the boundary conditions on Γ_F, the variational inequality

$$\int_\Omega \sigma^*_{ij}(\varepsilon^*_{ij} - \varepsilon_{ij})\, d\Omega - \int_\Omega p_i(u^*_i - u_i)\, d\Omega + \int_{\Gamma_S} k(u_{N+})(u^*_{N+} - u_{N+})\, d\Gamma - \int_{\Gamma_F} F_i(u^*_i - u_i)\, d\Gamma \geqslant 0 \quad \forall u^*_i \in X^*. \tag{7}$$

Applying the method of special variations, it has been proven that variational inequality (7) yields the equation of equilibrium and the boundary conditions on Γ_S and on Γ_F (see e.g. Panagiotopoulos, 1975). In this sense, the latter inequality completely characterizes the position of equilibrium of the body Ω. From the standpoint of mechanics, variational inequality (7) expresses the principle of virtual work in its inequality form for the body, having taken into account the unilateral boundary conditions. It has been also proven that at the position of equilibrium any solution of the variational inequality problem (7) minimizes over X^* the potential energy of the body given by the form

$$\Pi = \frac{1}{2}\int_\Omega \sigma_{ij}\varepsilon_{ij}\, d\Omega - \int_\Omega p_i u_i\, d\Omega + \int_{\Gamma_S} K(u_{N+})\, d\Gamma - \int_{\Gamma_F} F_i u_i\, d\Gamma, \tag{8}$$

where $K(\cdot)$, due to the monotonicity of $k(\cdot)$, is a convex function defined by the following integral:

$$K(\xi) = \int_0^\xi k(\xi)\, d\xi. \tag{9}$$

Conversely, it has been proven that any solution of the quadratic optimization problem (8) satisfies the variational inequality problem (7). A dual approach with respect to stresses leading to equivalent results can be also employed. In this case, a variational inequality problem expressing, from the standpoint of mechanics, the principle of complementary virtual work is formulated. The latter gives rise to a quadratic optimization problem of the complementary energy of the body Ω (see for example, Panagiotopoulos 1975, 1976, 1985).

2.2. *The discrete problem*

The present section deals with the mathematical description of the separation problem of bolted steel column-to-column connections in discrete form by applying the previously

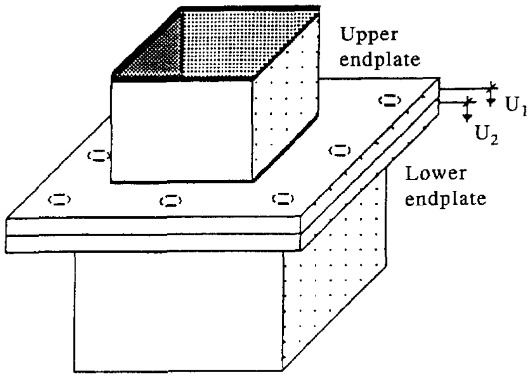

Fig. 2. A typical bolted steel column-to-column connection.

presented theory. A typical bolted steel connection is first considered (Fig. 2). In this connection, under certain loading conditions, the endplates tend to separate. In this case, construction practice and laboratory experience do also confirm that contact surfaces between joint members tend to separate. Indeed, the stress distribution along the bolts, which are symmetrically located about the axes of symmetry, is very uncertain. It is therefore, obvious that more elegant numerical approaches need to be applied for separation zones on the adjacent fronts of the connection, as well as stress distribution along the bolts, to be accurately defined.

In the sequel, a method for the numerical simulation of the structural response of such bolted steel splices is first presented. As has been previously stated, since on the detached regions between the contact fronts no reaction forces appear, whereas contact reactions appear on the active contact regions, the development of the separation phenomenon significantly affects the response of the steel connection. To begin with, the connection is discretized by means of an appropriate finite element scheme. In particular, plate elements are used to simulate the behaviour of endplates, whereas the separation conditions holding on the contact interface are realized by means of one-dimensional elastic contact elements (elastic couplers of infinitesimal length) connecting the adjacent nodes of the contact fronts. The mechanical behaviour of the contact elements simulating the possibility of separation of the adjacent nodes of the contact fronts are mathematically described, for instance for the ith contact element, by means of the following law:

$$\text{if } [u_z(i)] > 0 \quad \text{then} \quad R_z(i) = 0 \tag{10}$$

$$\text{if } [u_z(i)] = 0 \quad \text{then} \quad R_z(i) \geqslant 0, \tag{11}$$

where $[u_z(i)]$ denotes the relative displacements along the z-axis between the upper and lower splice plate, and $R_z(i)$ the respective reaction force. By means of relation (11), it is stated that if the region between the splice endplates connected by the ith spring are in contact, then the reaction force does exist on the contact region, whereas relation (10) expresses that if separation phenomena occur, reaction is equal to zero. It is also assumed that the response of the steel connection under investigation is not affected by any friction effects (frictionless type of contact). Assuming that the lower column flange can be considered rigid (i.e. exhibiting zero z-axis displacements), then the previous separation conditions (10),(11) can be put in the form

$$\text{if } u_z(i) > 0 \quad \text{then} \quad R_z(i) = 0 \tag{12}$$

$$\text{if } u_z(i) = 0 \quad \text{then} \quad R_z(i) \geqslant 0, \tag{13}$$

where $u_z(i)$ denotes the z-axis displacements of the splice endplate in the neighbourhood of the ith contact element. Assembling relations (12), (13) in matrix terms for all the m contact elements, the following linear complementarity problem (L.C.P.) is formulated:

$$\mathbf{u}_z \geqslant 0 \quad \mathbf{R}_z \geqslant 0 \quad \mathbf{R}_z^{\mathrm{T}}\mathbf{u}_z = 0, \tag{14}$$

where bold letters denote vectors and matrices, and superscript T denotes transposed vectors or matrices. This L.C.P. (14) completely describes in a quasi-static way the development of the separation phenomenon between the column endplates of the connection. We note also that the formulated L.C.P. (14) holds on this part of the boundary of the discretized steel connection, where unilateral contact conditions hold. Applying now the stiffness method to the simulation of the structural response of the discretized connection, the following matrix equation is obtained:

$$\mathbf{Ku} = \mathbf{P}, \tag{15}$$

where $\mathbf{K}$ is the stiffness matrix of the discretized connection, $\mathbf{u}$ the displacement vector for the whole structure, including vector $\mathbf{u}_z$, and $\mathbf{P}$ the load vector. As has been proposed by the authors (see e.g. Abdalla, 1988, and Baniotopoulos *et al.*, 1992), the problem of accurately defining the development of the separation zones between column endplates is completely described by the following quadratic programming problem (Q.P.P.):

$$\Pi(\mathbf{u}) = \min\left\{\tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{Ku}-\mathbf{P}^{\mathrm{T}}\mathbf{u} \mid \mathbf{Au} \leqslant \mathbf{b}\right\}, \tag{16}$$

where $\mathbf{A}$ (respectively $\mathbf{b}$) is an appropriately chosen transformation matrix [respectively vector describing the restrictions imposed by the inequalities (12, 13)]. The Q.P.P. (16) expresses, from the standpoint of mechanics, the principle of minimum potential energy for the steel connection at the state of equilibrium. The actual displacements of the connection plates caused by the external loading, as well as the active contact and the separation zones between the members of the connection, can be accurately defined by solving the formulated Q.P.P. (16). As has been previously noted for the continuous problem, a dual approach can also be employed for the treatment of the discrete problem. Such a dual approach equivalently gives rise to a quadratic programming problem of the same type, where stresses are the unknown variables appearing in the quadratic term and the constraints concern the equilibrium equation and the reaction forces appearing on the endplates, i.e.

$$\Pi^{C}(s) = \min\left\{\tfrac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{F}_0\mathbf{s}-\mathbf{s}^{\mathrm{T}}\mathbf{e}_0 \mid \mathbf{R}_z \geqslant 0, \quad \mathbf{Gs} = \mathbf{P}\right\}, \tag{17}$$

where $\mathbf{G}$ (respectively $\mathbf{F}_0$) is the equilibrium (respectively flexibility) matrix of the steel connection and $\mathbf{s}$ (respectively $\mathbf{e}_0$) the stress (respectively initial strain) vector. The Q.P.P. (17) expresses, from the standpoint of mechanics, the principle of minimum complementary energy for the steel connection at the state of equilibrium.

3. THE SENSITIVITY ANALYSIS PROBLEM

In order to obtain sensitivity analysis results for the structural response of the steel connection at hand, the unified approach for sensitivity analyses of unilateral problems for discrete and continuous structures proposed by Bendsøe *et al.* (1985) is now applied. This way, the variation of the structural behaviour of the connection subjected to variations of its material and geometrical characteristics or loading, being a critical factor in redesigning or optimizing the shape of the joint, can be completely described.

As previously mentioned, the appearance of the inequality constraints due to the unilateral contact phenomenon, leads to the formulation of an inherently nonlinear and nondifferentiable problem, where only directional sensitivities can be defined. Within a functional analysis framework, using modern analysis techniques combined to the minimum principles of potential and complementary energy of structures, sensitivities have been obtained in a general abstract setting. These sensitivity analysis results have also been extended to cover the analysis of discrete structures or structures numerically simulated by finite element models. In particular, sensitivity results have been obtained for structures with unilateral boundary conditions by first formulating and solving the quadratic optimization problem expressing from a mechanical point of view the principle of minimum potential or complementary energy for the discretized connection and then, relating the solution of the initial analysis problem to this one of a related quadratic optimization problem involving the same stiffness matrix, but different load term and different set of constraints.

We consider first the quadratic optimization problem (16) which describes the structural response of the steel connection written in the following form:

$$\Pi = \min\left\{\tfrac{1}{2}K^r_{ij}u_iu_j - P^r_iu_i \mid a_{ki}u_i \leqslant b_k \quad (i = 1,\ldots,n;\quad k = 1,\ldots,m)\right\}, \tag{18}$$

where the usual summation convention holds over the repeated matrices and K^r_{ij} is a family of symmetric matrices that are uniformly positive definite with respect to the design parameter r that belongs to the interval $[0, r_f$ with $r_j > 0$. Loading P^r_i and quantities a_{ki} and b_k that define the unilateral contact restrictions on the connection endplates take on each node a given value for $i = 1,\ldots,n$; $k = 1,\ldots,m$. Next we assume that the unilateral constraint set is nonempty and denote by u^r, the unique solution of problem (18). Then, the Kuhn–Tucker conditions, i.e. the necessary conditions for u^r to be the unique solution of problem (18), are written in the form

$$K^r_{ij}u_j - P^r_i + l^r_ka_{ki} = 0 \quad (i = 1,\ldots,n;\quad k = 1,\ldots,m) \tag{19}$$

and

$$l^r_k = 0 \quad \text{if} \quad a_{ki}u^r_i < b_k \quad (i = 1,\ldots,n;\quad k = 1,\ldots,m) \tag{20}$$

$$l^r_k \geqslant 0 \quad \text{if} \quad a_{ki}u^r_i = b_k \quad (i = 1,\ldots,n;\quad k = 1,\ldots,m), \tag{21}$$

where l^r_k are the respective Lagrange multipliers which express, from a mechanical point of view, the distribution of reactions on the active contact regions of the connection endplates having as consequence the development of prying action phenomena; the latter give rise to local separation phenomena between the two column endplates.

We define next the set $V(u^0)$ of the endplate nodes which are in contact [i.e. the set of active constraints of the Q.P.P. (18)]. The sets $V_0(u^0)$ and $V_1(u^0)$ corresponding to the sets of active constraints respectively with zero and nonzero Lagrange multipliers are defined as follows:

$$V(u^0) = \{k \mid a_{ki}u^0_i = b_k\}, \tag{22}$$

$$V_0(u^0) = \{k \in V(u^0) \mid l^r_k = 0\} \tag{23}$$

and

$$V_1(u^0) = \{k \in V(u^0) \mid l^r_k \neq 0\}. \tag{24}$$

By defining now the sets:

$$C = \{v \mid a_{ki}v^r_i \leqslant 0 \quad \text{for} \quad k \in V(u^0)\} \tag{25}$$

and

$$H = \{v \mid k^0_{ij} u^0_i v_j - P^0_i v_i = 0\} \tag{26}$$

and taking into account relation (19) that also holds for the basic (initial analysis) solution multiplied by v_i, i.e.

$$K^0_{ij} u^0_j v_i - P^0_i v_i + l^0_k a_{ki} v_i = 0 \quad (i = 1, \dots, n; \quad k = 1, \dots, m), \tag{27}$$

where superscripts 0 denote vectors and matrices corresponding to the basic solution of problem (18) for $r = r_0$, we obtain the set

$$Y = \{v \mid a_{ki} v_i \leqslant 0 \quad \text{for} \quad k \in V_0(u^0) \quad \text{and} \quad a_{ki} v_i = 0 \quad \text{for} \quad k \in V_1(u^0)\}. \tag{28}$$

By this mathematical formalism, possible contact and separation zones on the endplates are defined, excluding the possibility of appearance of a set of nodes where both z-axis reactions and displacements are equal to zero; these set of nodes would be points of irregularity in the solution of the sensitivity problem and would also lead to erroneous results.

A theorem already proved for sensitivity analysis problems (Bendsøe *et al.*, 1985) is next applied. Assume that

$$K'_{ij} = \lim \frac{(K^r_{ij} - K^0_{ij})}{r}$$

where

$$r \to 0^+ \quad \text{and} \quad P'_i = \lim \frac{(P^r_i - P^0_i)}{r} \quad \text{with } r \to 0^+.$$

Then, for r positive and small enough,

$$u^r = u^0 + ru' + O(r), \tag{29}$$

with

$$\frac{\|O(r)\|}{r \to 0}$$

(for $r \to 0$) and u' is the solution of the following quadratic programming problem:

$$\Pi = \min \{\tfrac{1}{2} K^0_{ij} v_i v_j - P'_i v_i + K'_{ij} u^0_i v_j \mid$$
$$a_{ki} v_i \leqslant b_k \quad \text{for} \quad k \in V_0(u^0) \quad \text{and} \quad a_{ki} v_i = b_k \quad \text{for} \quad k \in V_1(u^0)\}. \tag{30}$$

In the case that $V(u^0) = V_1(u^0)$, relation (29) holds in the following form:

$$u^r = u^0 \pm ru'_\pm \mp O(r), \tag{31}$$

for r belonging to the open interval $(-r_f, +r_f)$ and r_f positive and small enough. Here $u'_\pm$ is the actual solution of the problem

$$\Pi = \min \{\tfrac{1}{2} K^0_{ij} v_i v_j \mp P'_i v_i \pm K'_{ij} u^0_i v_j \mid$$
$$a_{ki} v_i \leqslant b_k \quad \text{for } k \in V_0(u^0) \quad \text{and} \quad a_{ki} v_i = b_k \quad \text{for } k \in V_1(u^0)\} \tag{32}$$

and in general $u'_+ \neq -u'_-$ confirming that the problem is not differentiable, but only directionally differentiable. Note that the so-called first-order necessary condition recently proved for the problem at hand reads

$$k'_{ij}u^0_j + k^0_{ij}u'_j - f'_r = l^{a'}_k b_{ki} \quad \text{for} \quad i = 1, \ldots, k; \quad k \in V(u^0), \tag{33}$$

where

$$l^a_k = \lim \frac{(l^r_k - l^0_k)}{r} \quad \text{for } r \to 0^+ \quad \text{and} \quad k \in V(u^0).$$

Since problem (30) involves the same stiffness matrix with the initial analysis problem (18), sensitivities can be calculated by applying the same solution method applied to the solution of the initial quadratic programming problem (18). Obviously, in the case that vectors a_{ki} are linearly independent, Lagrange multipliers

$$l^{a'}_k = \frac{l^r_k - l^0_k}{r} \quad \text{for} \quad r \to 0^+$$

of problem (30) coincide to the reaction forces on the activated contact nodes of the column endplates being uniquely determined.

4. ON THE ALGORITHMIC TREATMENT OF THE FORMULATED PROBLEMS

In the previous paragraphs the sensitivity analysis problem of steel bolted column connections has been formulated as a quadratic optimization problem which involves the same quadratic term with the initial analysis problem. For this reason, since the stiffness matrix is the same in both the analysis and the sensitivity problem, the stiffness matrix has only to be assembled once. As is obvious, the same solution method can be applied to the numerical treatment of both problems (18) and (30) and, in particular, the Hildreth–d'Esopo algorithm which is briefly discussed in the sequel for the analysis problem.

For the numerical treatment of the previously formulated Q.P.Ps, the Hildreth–d'Esopo algorithm is applied, being a typical iterative procedure and having the advantage of being easily programmable and computationally efficient (see e.g. Abdalla and Stavroulakis, 1989). As is well known, the Kuhn–Tucker optimality conditions for the Q.P.P. (16) can be written in the following form:

$$\mathbf{Au} + \mathbf{y} = \mathbf{b} \tag{34}$$

$$\mathbf{Ku} + \mathbf{A}^{\mathrm{T}}\mathbf{f} = \mathbf{P} \tag{35}$$

$$\mathbf{y} \geqslant 0, \quad \mathbf{f} \geqslant 0, \quad \mathbf{y}^{\mathrm{T}}\mathbf{f} = 0, \tag{36}$$

where $\mathbf{y}$ is a vector corresponding to the unilateral constraints of the problem and $\mathbf{f}$ the vector of reactions on the same constraints. Solving eqn (35) with respect to $\mathbf{u}$, the following relation is obtained:

$$\mathbf{u} = -\mathbf{K}^{-1}(\mathbf{A}^{\mathrm{T}}\mathbf{f} - \mathbf{P}), \tag{37}$$

and then, putting

$$\mathbf{h} = -\mathbf{AK}^{-1}\mathbf{P} + \mathbf{b} \tag{38}$$

and

$$\mathbf{F} = \tfrac{1}{2}\mathbf{A}\mathbf{K}^{-1}\mathbf{A}^{T}, \tag{39}$$

relations (34)–(36) can be written as follows,

$$2\mathbf{F}\mathbf{f} - \mathbf{y} = -\mathbf{h} \tag{40}$$

$$\mathbf{y} \geqslant 0, \quad \mathbf{f} \geqslant 0, \quad \mathbf{y}^{T}\mathbf{f} = 0. \tag{41}$$

The latter relations constitute the Kuhn–Tucker optimality conditions for the following Q.P.P.:

$$\Pi(\mathbf{f}) = \min\left\{\tfrac{1}{2}\mathbf{f}^{T}\mathbf{F}\mathbf{f} + \mathbf{h}^{T}\mathbf{f} \mid \mathbf{f} \geqslant 0\right\}, \tag{42}$$

where matrix $\mathbf{F}$ is a flexibility matrix defined by eqn (39) relating contact forces to the corresponding unilateral contact displacements. When the solution of problem (16) exists, then problem (42) does also have a solution and this is unique. The Q.P.P. (42) can numerically be treated by means of the Gauss–Seidel method. During the iterative steps $p = 0, 1, 2, \ldots$, the following iterative values are considered:

$$f_i^{p+1} = \max\{0, \omega_i^{p+i}\}, \tag{43}$$

where

$$\omega_i^{p+i} = -\frac{1}{g_{ii}}\left(\sum_{j=1}^{i-1} g_{ij} l_j^{p+1} + \frac{h_i}{2} + \sum_{j=j+1}^{m} g_{ij} l_j^{p}\right) \quad i = 1, 2, \ldots m \tag{44}$$

and m is the number of constraints of the problem. Iterations stop when the computed contact reactions pass the imposed accuracy criteria, i.e. when

$$\| f_j^{p} - f_j^{p+1} \| \leqslant \varepsilon, \tag{45}$$

where the symbol $\| \cdot \|$ denotes an appropriately defined norm.

5. NUMERICAL APPLICATIONS

The code BOLT-1 based on the Hildreth–d'Esopo quadratic programming algorithm has been developed and, after the application of an appropriate discretization scheme, the following numerical examples have been numerically investigated on a Hewlett-Packard 750 RISC Workstation.

The previously presented method has been applied in order to obtain sensitivity analysis results with respect to splice plate thickness for the design problem of two steel bolted column-to-column connections. In both numerical examples, modulus of elasticity and Poisson's ratio for the material of the connections have been taken, respectively, as $E = 2.1\ 10^{7}\ N\ \mathrm{cm}^{-2}$ and $\nu = 0.30$. The first example deals with a steel splice of two columns with dimensions $60 \times 60 \times 5$ mm (upper) and $150 \times 150 \times 5$ mm (lower). The dimensions of both splice endplates are $300 \times 300 \times d$ mm, where d is the thickness of the plate. Considering as basic column endplate thickness $d = 12$ mm, displacements for both the upper and lower splice plates are calculated by solving the formulated problem (16) by the Hildreth–d'Esopo algorithm, and the regions of separation and contact are defined accurately (Figs 3 and 4).

Formulating now problem (30) and numerically treating it with the same algorithm, sensitivity analysis results with respect to the thickness of the connection plates are obtained (Figs 5 and 6).

The column-to-column steel bolted splice design problem for orthogonal columns ($250 \times 150 \times 6.3$ and $450 \times 150 \times 10$ mm) and orthogonal endplates ($450 \times 700 \times d$ mm) was investigated next. The basic solution for endplate thickness $d = 14$ mm is depicted in Figs

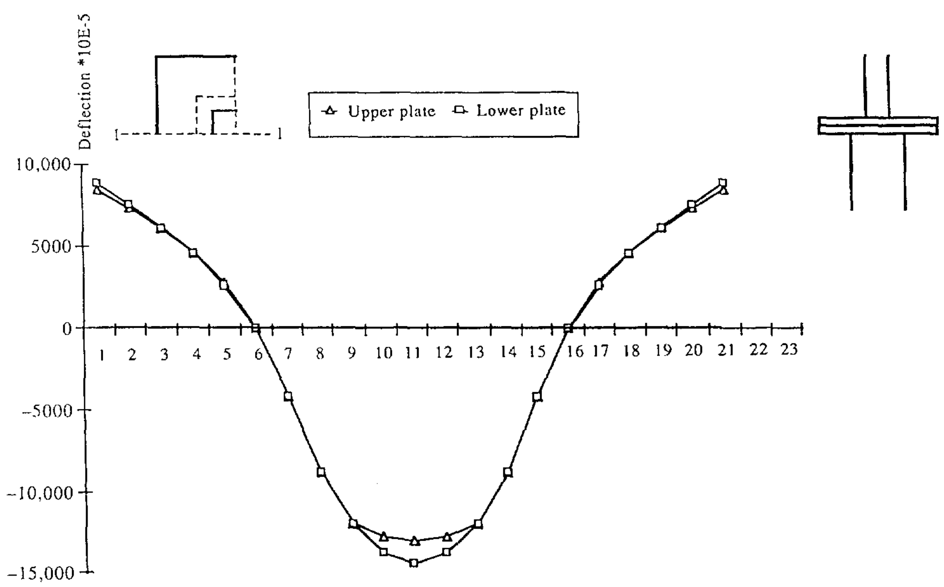

Fig. 3. Displacements of the upper and lower connection plates along cross-section 1–1 (square column and plate section, thickness $d = 12$ mm).

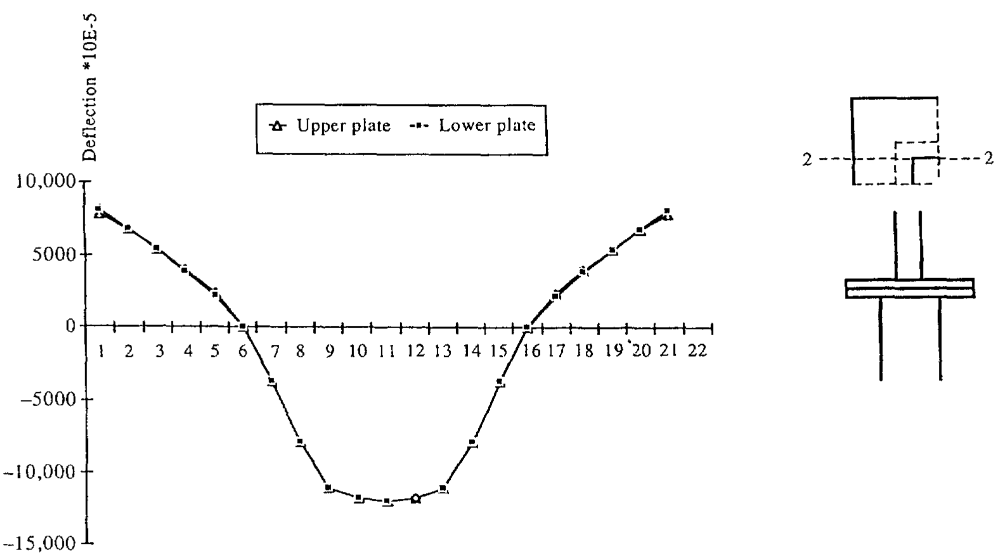

Fig. 4. Displacements of the upper and lower connection plates along cross-section 2–2 (square column and plate section, thickness $d = 12$ mm).

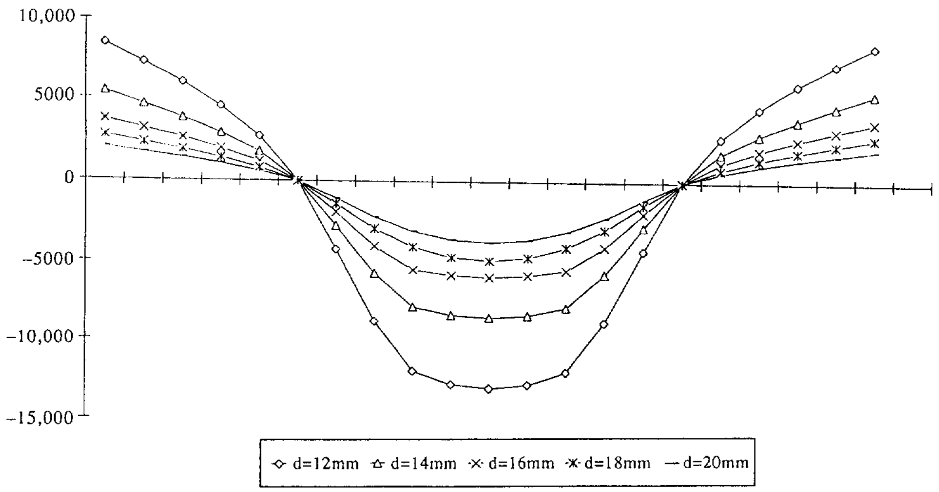

Fig. 5. Sensitivity analysis results with respect to connection plate thickness for the displacements of the upper splice plate (square section, cross-section 1–1).

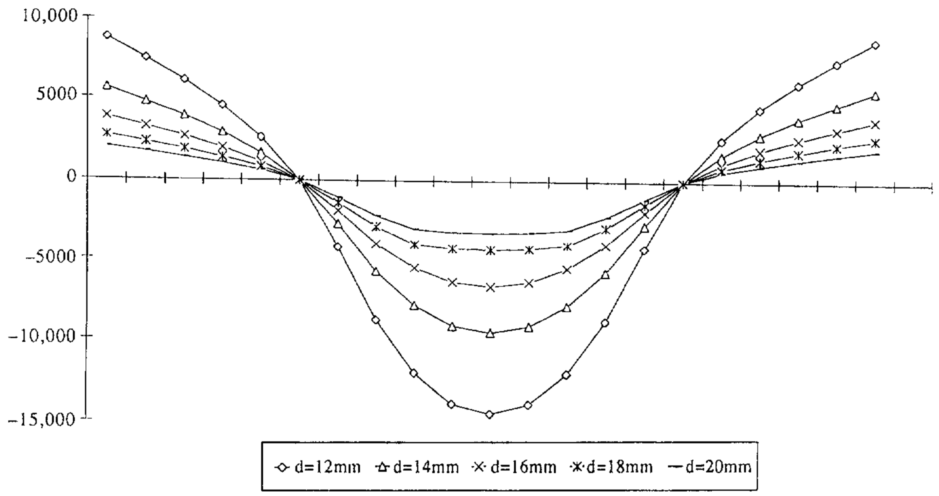

Fig. 6. Sensitivity analysis results with respect to connection plate thickness for the displacements of the lower splice plate (square section, cross-section 1–1).

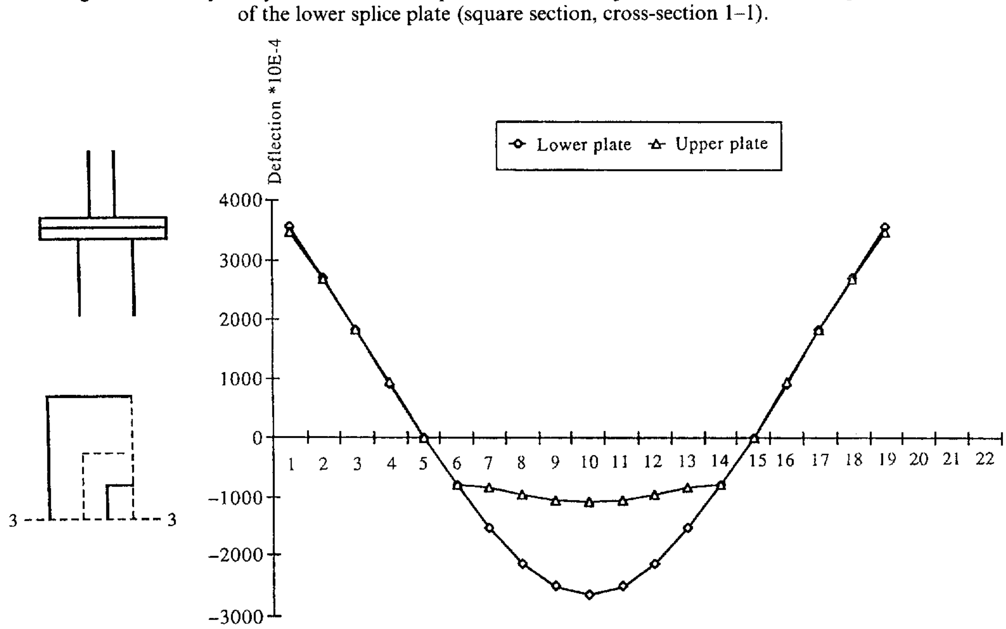

Fig. 7. Displacements of the upper and lower connection plates along cross-section 3–3 (orthogonal column and plate section, thickness $d = 14$ mm).

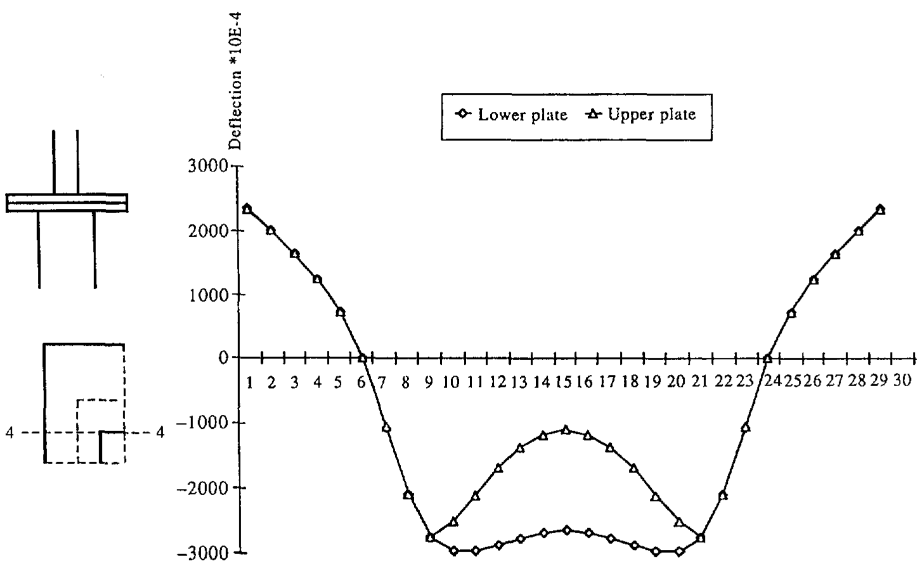

Fig. 8. Displacements of the upper and lower connection plates along cross-section 4–4 (orthogonal column and plate section, thickness $d = 14$ mm).

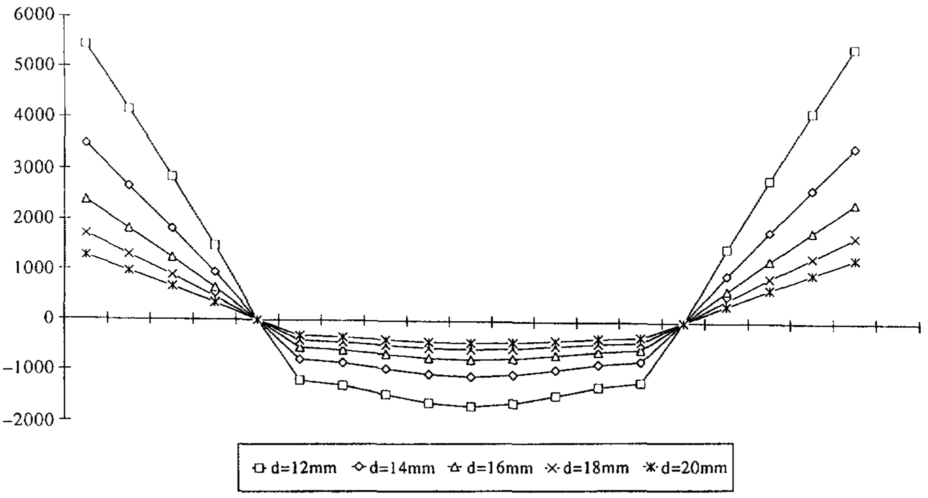

Fig. 9. Sensitivity analysis results with respect to connection plate thickness for the displacements of the upper splice plate (orthogonal section, cross-section 3–3).

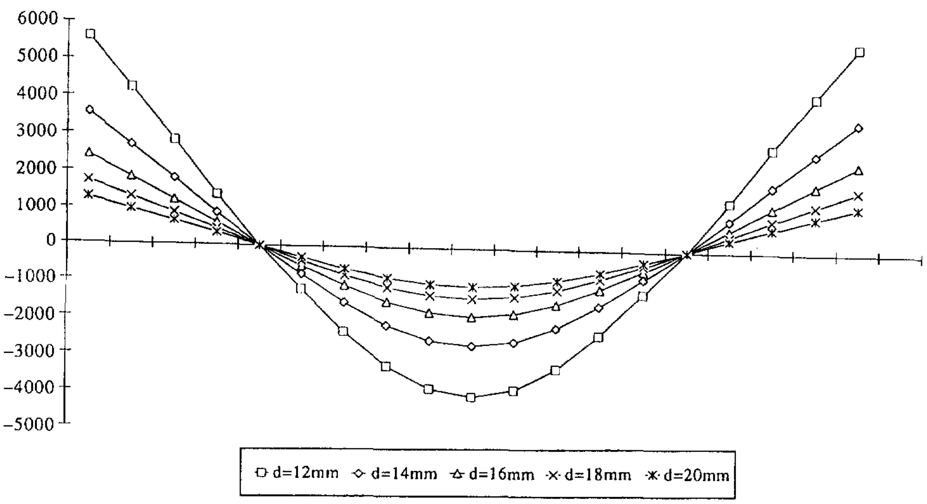

Fig. 10. Sensitivity analysis results with respect to connection plate thickness for the displacements of the lower splice plate (orthogonal section, cross-section 3–3).

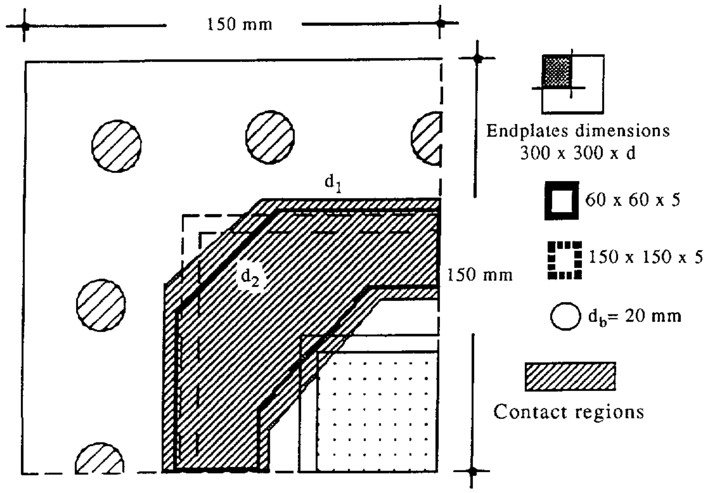

Fig. 11. Variation of the active contact and separation zones on the adjacent fronts with respect to the variation of thickness (square section).

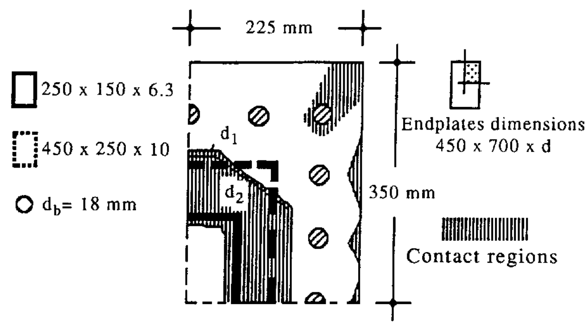

Fig. 12. Variation of the active contact and separation zones on the adjacent fronts with respect to the variation of thickness (orthogonal section).

7 and 8, where separation and active contact zones have been defined. Sensitivity results with respect to the connection plate thickness concerning the development of regions of separation on the splice plates are depicted in Figs 9 and 10. In Figs 11 and 12, the variation of the active contact and separation zones on the contact interface with respect to the thickness variation of the splice connections for both the previously investigated problems is depicted.

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